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Boundary effects on the dispersion force between oscillators

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Abstract. A general theory is developed to study the effect of boundaries on the dispersion interaction between molecules in a bounded region. By way of illustration the method is applied to the study of the interaction between two oscillators located between two conducting plates. When the oscillators are separated by a distance larger than the plate separation the force law becomes considerably different from London and Casimir–Polder results.

1. Introduction

When two molecules are placed in a bounded region, such as a box or a channel, the dispersion force between them is expected to be different from what it is when they are in free space. The difference arises from the dependence of the structure of the modes of the electromagnetic field in the region on the boundary conditions, a point that has already been noted in the quantum electrodynamics of charged particles in such situations (Barton 1970). The simplest formulation of the problem can be made in semi-classical terms (Mahanty and Ninham 1972, to be referred to as I), in which the molecules are regarded as point-dipole oscillators coupled to each other through the electromagnetic field. The dispersion interaction is the difference between the zero-point energy of the coupled oscillator system and that of two single oscillators. The effect of the boundary enters through the structure of the Green functions of the electromagnetic field that determine the coupling between the oscillators.

To see this explicitly, we start with the equations of motion of two isotropic oscillators of natural frequency ω_0, charge $(-e)$ and mass m, in time independent (Fourier transform) form,

$$m(\omega_0^2-\omega^2)\boldsymbol{u}_j(\omega) = \frac{\mathrm{i}\omega e}{c}\mathscr{A}(\boldsymbol{R}_j, \omega)+e\nabla\phi(\boldsymbol{R}_j, \omega); \qquad j = 1, 2. \tag{1}$$

Here $\boldsymbol{u}_j$ is the displacement from equilibrium of the jth oscillator, and $\boldsymbol{R}_j$ is the coordinate of its equilibrium position, as also that of the core positive charge $(+e)$ which is assumed to be stationary. The time independent equations of motion of the vector and scalar potentials $\mathscr{A}$ and ϕ are (in Coulomb gauge),

$$\left(\nabla^2+\frac{\omega^2}{c^2}\right)\mathscr{A}(\boldsymbol{r}, \omega) = \frac{\mathrm{i}\omega}{c}\nabla\phi+\frac{4\pi\mathrm{i}\omega e}{c}\sum_j \boldsymbol{u}_j(\omega)\,\delta(\boldsymbol{r}, \boldsymbol{R}_j), \tag{2a}$$

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with

$$\nabla \cdot \mathscr{A} = 0, \tag{2b}$$

and

$$\nabla^2 \phi = 4\pi e \sum_j \{(\nabla_{R_j} \delta(\boldsymbol{r}, \boldsymbol{R}_j)) \cdot \boldsymbol{u}_j(\omega)\}. \tag{3}$$

The vector and scalar potentials $\mathscr{A}$ and ϕ can be eliminated in equation (1) by solving for $\mathscr{A}$ and ϕ from equations (2) and (3), and we then obtain the secular determinant for the coupled oscillator system in the form,

$$D_{12}(\omega) = \begin{vmatrix} m(\omega_0^2 - \omega^2)\mathbf{I} + 4\pi e^2 \mathscr{G}(\boldsymbol{R}_1, \boldsymbol{R}_1; \omega) & 4\pi e^2 \mathscr{G}(\boldsymbol{R}_1, \boldsymbol{R}_2; \omega) \\ 4\pi e^2 \mathscr{G}(\boldsymbol{R}_2, \boldsymbol{R}_1; \omega) & m(\omega_0^2 - \omega^2)\mathbf{I} + 4\pi e^2 \mathscr{G}(\boldsymbol{R}_2, \boldsymbol{R}_2; \omega) \end{vmatrix}. \tag{4}$$

Here, the diadic Green function $\mathscr{G}(\boldsymbol{r}, \boldsymbol{r}'; \omega)$ is given by,

$$\mathscr{G}(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \frac{\omega^2}{c^2} \mathbf{G}^{(2)}(\boldsymbol{r}, \boldsymbol{r}'; \omega) - \nabla\nabla' G^{(1)}(\boldsymbol{r}, \boldsymbol{r}'), \tag{5}$$

where $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$ is the Green function of the equation

$$\nabla^2 \phi = 0, \tag{6}$$

and $\mathbf{G}^{(2)}(\boldsymbol{r}, \boldsymbol{r}'; \omega)$ is the diadic Green function of the equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \mathscr{A} = 0 \tag{7}$$

with the appropriate boundary conditions. $(\nabla\nabla')$ is the diadic operator formed out of the gradient operators operating on the unprimed and primed coordinates.

The dispersion interaction energy can be evaluated by the formula (see I),

$$E(R) = -\frac{\hbar}{4\pi \mathrm{i}} \int_{\mathrm{i}\infty}^{-\mathrm{i}\infty} \ln\left(\frac{D_{12}(\omega)}{D_1(\omega) D_2(\omega)}\right) \mathrm{d}\omega, \tag{8}$$

where $R = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$, and

$$D_j(\omega) = |m(\omega_0^2 - \omega^2)\mathbf{I} + 4\pi e^2 \mathscr{G}(\boldsymbol{R}_j, \boldsymbol{R}_j; \omega)|. \tag{9}$$

To order (e^4) the formula for $E(R)$ becomes, after some simplification,

$$E(R) \simeq -\frac{8\pi\hbar e^4}{m^2} \left(\int_0^\infty \frac{\mathrm{d}\xi}{(\omega_0^2 + \xi^2)^2} \mathrm{Tr}(\mathscr{G}(\boldsymbol{R}_1, \boldsymbol{R}_2; -\mathrm{i}\xi) \mathscr{G}(\boldsymbol{R}_2, \boldsymbol{R}_1; -\mathrm{i}\xi)) \right). \tag{10}$$

The use of Coulomb gauge separates out the non-retarded form of $\mathscr{G}$ in equation (5) explicitly, so that in the non-retarded limit, with $c \to \infty$, equation (10) becomes,

$$E(R) \simeq -\frac{2\pi^2 \hbar e^4}{m^2 \omega_0^3} \mathrm{Tr}\{(\nabla_{R_1} \nabla_{R_2} G^{(1)}(\boldsymbol{R}_1, \boldsymbol{R}_2))(\nabla_{R_2} \nabla_{R_1} G^{(1)}(\boldsymbol{R}_2, \boldsymbol{R}_1))\}. \tag{11}$$

Our analysis will be based on equations (10) and (11).

2. The structure of the Green functions

The main difference between the structure of the Green functions in free space and in bounded regions arises out of the discretization of some or all the modes of the electromagnetic field in the bounded region. The precise boundary conditions to be used depend on the physical properties of the boundary, such as its dielectric constant or conductivity.

The general theory of the Green functions $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$ and $\mathbf{G}^{(2)}(\boldsymbol{r}, \boldsymbol{r}'; \omega)$ is well known (Morse and Feshbach 1953) and can be adapted suitably to be applicable to this problem. $G^{(1)}$ can be constructed out of the solutions of the scalar equation,

$$\nabla^2 \chi_\lambda(\boldsymbol{r}) = \lambda \chi_\lambda(\boldsymbol{r}), \tag{12}$$

where $\chi_\lambda(\boldsymbol{r})$ is suitably normalized and satisfies the same boundary conditions as ϕ. Then,

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \sum_\lambda \frac{\chi_\lambda(\boldsymbol{r})\chi_\lambda^*(\boldsymbol{r}')}{\lambda}. \tag{13}$$

The diadic Green function $\mathbf{G}^{(2)}$ is obtained from the divergence-free vector solutions of equation (7), or equivalently, of the equation,

$$\nabla^2 \boldsymbol{F}_\lambda(\boldsymbol{r}) = \lambda \boldsymbol{F}_\lambda(\boldsymbol{r}). \tag{14}$$

The two independent divergence-free solutions of this equation can be written in the form,

$$\boldsymbol{M}_\lambda(\boldsymbol{r}) = \nabla \times (\boldsymbol{a}\psi_\lambda), \tag{15a}$$

$$\boldsymbol{N}_\lambda(\boldsymbol{r}) = \frac{1}{k}[\nabla \times \{\nabla \times (\boldsymbol{a}\psi'_\lambda)\}] \tag{15b}$$

where $\psi_\lambda, \psi'_\lambda$ satisfy equation (12), and k (having the dimension of a wavenumber), the direction of the unit vector $\boldsymbol{a}$ and the relative magnitudes of $\boldsymbol{M}_\lambda$ and $\boldsymbol{N}_\lambda$ are suitably adjusted to make the function $\boldsymbol{F}_\lambda = \boldsymbol{M}_\lambda + \boldsymbol{N}_\lambda$ satisfy the right boundary conditions. Both $\boldsymbol{M}_\lambda$ and $\boldsymbol{N}_\lambda$ can be normalized as,

$$\int (\boldsymbol{M}_\lambda^* \cdot \boldsymbol{M}_{\lambda'})\,\mathrm{d}^3r = \int (\boldsymbol{N}_\lambda^* \cdot \boldsymbol{N}_{\lambda'})\,\mathrm{d}^3r = \Lambda_\lambda \delta_{\lambda\lambda'}. \tag{16}$$

Then

$$\mathbf{G}^{(2)}(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \sum_\lambda \frac{1}{\Lambda_\lambda(\omega^2/c^2 + \lambda)}(\boldsymbol{M}_\lambda(\boldsymbol{r})\boldsymbol{M}_\lambda^*(\boldsymbol{r}') + \boldsymbol{N}_\lambda(\boldsymbol{r})\boldsymbol{N}_\lambda^*(\boldsymbol{r}')), \tag{17}$$

where $\boldsymbol{M}_\lambda\boldsymbol{M}_\lambda^*$ and $\boldsymbol{N}_\lambda\boldsymbol{N}_\lambda^*$ are the diadics formed out of the vectors $\boldsymbol{M}_\lambda$ and $\boldsymbol{N}_\lambda$.

3. The oscillators between conducting plates

We shall apply the above formalism to the situation in which the two oscillators are located between two parallel perfectly conducting plates with separation L. Here the tangential component of the electric field and the electrostatic potential vanish on the plates. Choosing a coordinate system in which the origin is on one of the plates and the z axis normal to the plates, and using the boundary conditions,

$$\mathscr{A}_x = 0 = \mathscr{A}_y; \qquad \frac{\partial \mathscr{A}_z}{\partial z} = 0 \qquad \text{and} \qquad \phi = 0 \qquad \text{at } z = 0, L, \tag{18}$$

we obtain the following expressions for the Green functions:

$$G^{(1)}(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{2\pi^2 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \times \int_{-\infty}^{\infty}\int \frac{\mathrm{d}k_1\,\mathrm{d}k_2\exp[\mathrm{i}\{k_1(x-x')+k_2(y-y')\}]}{k_n^2}, \tag{19}$$

$$\begin{aligned}\mathbf{G}^{(2)}(\boldsymbol{r},\boldsymbol{r}';\omega) = {} & \sum_{n=0}^{\infty}\frac{1}{\epsilon_n(2\pi^2 L)}\int_{-\infty}^{\infty}\int \frac{\mathrm{d}k_1\,\mathrm{d}k_2\exp[\mathrm{i}\{k_1(x-x')+k_2(y-y')\}]}{(\omega^2/c^2-k_n^2)\{1-(n^2\pi^2/k_n^2L^2)\}} \\ & \times\Bigg\{\frac{1}{k_n^2}(\boldsymbol{a}_1k_2-\boldsymbol{a}_2k_1)(\boldsymbol{a}_1k_2-\boldsymbol{a}_2k_1)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\ & +\frac{1}{k_n^4}\left(\frac{n\pi}{L}\right)^2(\boldsymbol{a}_1k_1+\boldsymbol{a}_2k_2)(\boldsymbol{a}_1k_1+\boldsymbol{a}_2k_2)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\ & -\frac{\mathrm{i}}{k_n^4}\left(\frac{n\pi}{L}\right)(k_1^2+k_2^2)(\boldsymbol{a}_1k_1+\boldsymbol{a}_2k_2)(\boldsymbol{a}_3)\sin\left(\frac{n\pi z}{L}\right)\cos\left(\frac{n\pi z'}{L}\right) \\ & +\frac{\mathrm{i}}{k_n^4}\left(\frac{n\pi}{L}\right)(k_1^2+k_2^2)(\boldsymbol{a}_3)(\boldsymbol{a}_1k_1+\boldsymbol{a}_2k_2)\cos\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \\ & +\frac{1}{k_n^4}(k_1^2+k_2^2)^2(\boldsymbol{a}_3)(\boldsymbol{a}_3)\cos\left(\frac{n\pi z}{L}\right)\cos\left(\frac{n\pi z'}{L}\right)\Bigg\}.\end{aligned} \tag{20}$$

Here

$$k_n^2 = k_1^2+k_2^2+\left(\frac{n^2\pi^2}{L^2}\right);$$

$$\epsilon_n\begin{cases}=2 & \text{for } n=0,\\ =1 & \text{for } n\neq 0,\end{cases}$$

and $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are the unit vectors in the x, y and z directions. Expressions of the sort $(\boldsymbol{a}_\alpha k_\beta-\boldsymbol{a}_\gamma k_\sigma)(\boldsymbol{a}_\tau k_\eta)$ represent the diadic formed out of the vectors $(\boldsymbol{a}_\alpha k_\beta-\boldsymbol{a}_\gamma k_\sigma)$ and $(\boldsymbol{a}_\tau k_\eta)$.

From equations (19), (20) and (5) the explicit form of $\mathscr{G}(\boldsymbol{r},\boldsymbol{r}';\omega)$ can be written as,

$$\mathscr{G}(\boldsymbol{r},\boldsymbol{r}';\omega) = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33}\end{bmatrix}, \tag{21}$$

where

$$T_{\alpha\beta} = \left(\frac{\omega^2}{c^2}\delta_{\alpha\beta}-\frac{\partial^2}{\partial r_\alpha \partial r'_\beta}\right)g_1(\boldsymbol{r},\boldsymbol{r}';\omega)+\frac{\omega^2}{c^2}\delta_{\alpha 3}\,\delta_{\beta 3}g_2(\boldsymbol{r},\boldsymbol{r}';\omega); \tag{22a}$$

$$g_1(\boldsymbol{r},\boldsymbol{r}';\omega) = \frac{1}{2\pi^2 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_1\,\mathrm{d}k_2\exp[\mathrm{i}\{k_1(x-x')+k_2(y-y')\}]}{\omega^2/c^2-k_n^2}; \tag{22b}$$

$$g_2(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \frac{1}{2\pi^2 L}\sum_{n=0}^{\infty} \cos\left(\frac{n\pi(z+z')}{L}\right) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}k_1\,\mathrm{d}k_2 \exp[\mathrm{i}\{k_1(x-x')+k_2(y-y')\}]}{\epsilon_n(\omega^2/c^2-k_n^2)}. \tag{22c}$$

For notational convenience we shall define a variable $\rho = \{(x-x')^2+(y-y')^2\}^{1/2}$ and obtain the dispersion interaction between the oscillators when the distance $\rho_{12} = \{(x_1-x_2)^2+(y_1-y_2)^2\}^{1/2}$ satisfies the condition $\rho_{12} \gg L$, and when $L \gg R$.

3.1. Interaction in the non-retarded limit

This limit, obtained for $c \to \infty$ corresponds to the situation when both L and R (and hence ρ) are much less than the characteristic wavelength $\lambda_0 = 2\pi c/\omega_0$. We shall use equation (11). When $\rho \gg L$ an appropriate form of $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$ is (see appendix),

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{\pi L}\sum_{n=1}^{\infty} K_0\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right). \tag{23}$$

For large (ρ/L), it is sufficient to retain only the $(n = 1)$ term and use the asymptotic form for $K_0(\pi\rho/L)$. Substituting this $G^{(1)}$ in equation (11) we get,

$$E(R) \simeq -\frac{2\pi^5 \hbar e^4}{m^2\omega_0^3 L^6}\left(\frac{\exp(-2\pi\rho_{12}/L)}{2\pi\rho_{12}/L}\right)\cos^2\left(\frac{\pi(z_1+z_2)}{L}\right). \tag{24}$$

For large L, an appropriate form of $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$ is,

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{4\pi}\int_0^{\infty} \frac{\mathrm{d}\kappa J_0(\kappa\rho)}{\sinh(\kappa L)}[\cosh\{\kappa(L-|z-z'|)\} - \cosh\{\kappa(L-|z+z'|)\}], \tag{25}$$

where $\kappa = (k_1^2+k_2^2)^{1/2}$. In the appendix the form of $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$ for a number of cases corresponding to different relative values of z, z' and L are given. Using these in equation (11) we get the following forms for $E(R)$ for those cases.

For $(z_1+z_2-L) \ll L$, that is, when the oscillators are nearly in the middle of the two conducting planes,

$$E(R) \simeq E(R)_{\mathrm{London}}\left\{1+\tfrac{5}{6}\zeta(3)\left(\frac{R}{L}\right)^3\left(\frac{3(z_1-z_2)^2}{R^2}-1\right)\right\} \tag{26a}$$

where $E(R)_{\mathrm{London}} = -3\hbar e^4/4m^2\omega_0^3 R^6$ is the non-retarded London interaction between the oscillators in free space. It may be noted that on the medial plane, that is, when $z_1 = z_2$, the interaction is diminished from the London result for free space. But when $|z_1-z_2| > (\rho_{12}/\sqrt{2})$ the interaction is enhanced from the free space value.

For $(z_1+z_2) \ll L$, that is, the oscillators are close to one of the conducting planes,

$$E(R) \simeq E(R)_{\mathrm{London}}\left\{\frac{2}{3}\left(1-\zeta(3)\frac{\rho_{12}^3}{L^3}\right)\right\}, \tag{26b}$$

which implies a reduction from the London value by a factor $\frac{2}{3}$.

For $(z_1+z_2) \sim L$, that is, the oscillators are placed one near each conducting plane, the corresponding result is

$$E(R) \simeq E(R)_{\text{London}}\left\{\left(\frac{R}{L}\right)^6 \frac{(\frac{7}{8}\zeta(3))^2}{6} + \dots\right\}, \tag{26c}$$

where the neglected terms are $O(\rho_{12}/L)^2$ and $O\{(|z_1 \pm z_2| - L)/L\}^2$.

3.2. Interaction in the retarded region

In this region we have to use the Green function of equation (21). Again we consider the two cases, $\rho_{12} \gg L$ and $L \gg R$. For $\rho \gg L$ the appropriate form of the Green function $g_1(\boldsymbol{r}, \boldsymbol{r}'; \omega)$ of equation (22) is,

$$g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = -\frac{1}{\pi L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) K_0\left[\rho\left\{\frac{\xi^2}{c^2} + \left(\frac{n\pi}{L}\right)^2\right\}^{1/2}\right], \tag{27}$$

with a similar expression for $g_2(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi)$. The algebra involved in using these Green functions in equations (21) and (10) is tedious but elementary, and we find,

$$E(R) \simeq -\frac{192}{\pi}\frac{\hbar c e^4}{m^2\omega_0^4}\frac{1}{L^2\rho_{12}^5}. \tag{28}$$

This is a curious result, since it represents an enhancement from the Casimir–Polder result for free space (Casimir and Polder 1948).

For the case $R \ll L$ the required manipulations are very much more complicated and we shall simply quote the results. The method of their derivation is given in the appendix.

For $(R\omega_0/c) \ll 1$, and $L \ll \lambda_0$ we have,

$$E(R) \simeq E(R)_{\text{London}}\left[1 + \left(\frac{3(z_1-z_2)^2}{R^2} - 1\right)\frac{2}{3\pi}\left(\frac{c}{\omega_0 L}\right)\left\{\left(\frac{R}{L}\right)^3\zeta(4) + O\left(\frac{R}{L}\right)^5\right\}\dots\right]. \tag{29}$$

For $(R\omega_0/c) \gg 1$, and $L \gg \lambda_0$ we have,

$$E(R) \simeq E(R)_{\text{C—P}}\left\{1 + \left(\frac{3(z_1-z_2)^2}{R^2} - 1\right)\frac{2}{69}\left(\frac{R}{L}\right)^4\zeta(4) + O\left(\frac{R}{L}\right)^6\right\}, \tag{30}$$

where

$$E(R)_{\text{C—P}} = -\frac{23\hbar c}{4\pi R^7}\frac{e^4}{m^2\omega_0^4}$$

is the retarded interaction result of Casimir and Polder (1948). Here also, as in equation (26*a*), the interaction is reduced for $|z_1 - z_2| < (\rho_{12}/\sqrt{2})$ and enhanced for $|z_1 - z_2| > (\rho_{12}/\sqrt{2})$. From equation (29) one can recover the London result for $L \to \infty$, but not for $c \to \infty$ with fixed L, as the condition under which the expansion is derived is for $L \ll \lambda_0$. The results for $c \to \infty$ with fixed L are already given in equation (26).

4. Conclusion

While the method developed here has been illustrated in the specific problem of the interaction of oscillators between conducting plates, it can be extended to study the interaction between molecules confined within arbitrarily shaped containers with dielectric boundaries. It is clear from the above analysis that the dispersion force changes markedly in bounded regions—in a narrow channel there is considerable reduction from the free space value in the non-retarded case, and increase in the retarded case.

For the problem of resonance energy transfer between two molecules in a bounded region, a similar semiclassical formulation can be developed. The effect of containment of two molecules in a channel is a marked enhancement in the process of resonance energy transfer (Mahanty and Ninham 1973).

Appendix

We have used several forms for the Green functions and indicate the method of derivation of the results given in the text.

Consider the function g_1, defined by equation (22*b*). We have (in the notation of § 3),

$$-g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = \frac{1}{2\pi^2 L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \int_0^\infty \kappa\, \mathrm{d}\kappa \int_0^{2\pi} \frac{\mathrm{d}\theta \exp(\mathrm{i}\kappa\rho\cos\theta)}{\xi^2/c^2+\kappa^2+n^2\pi^2/L^2}. \quad \text{(A.1)}$$

If we perform the κ and θ integration according to the formula,

$$\int_0^\infty \frac{\kappa\,\mathrm{d}\kappa}{\xi^2/c^2+n^2\pi^2/L^2+\kappa^2}\int_0^{2\pi}\mathrm{d}\theta\exp(\mathrm{i}\kappa\rho\cos\theta)$$
$$= 2\pi\int_0^\infty \frac{\kappa\,\mathrm{d}\kappa J_0(\kappa\rho)}{\xi^2/c^2+n^2\pi^2/L^2+\kappa^2} = 2\pi K_0\left\{\rho\left(\frac{\xi^2}{c^2}+\frac{n^2\pi^2}{L^2}\right)^{1/2}\right\}, \quad \text{(A.2)}$$

we obtain equation (27). For $c\to\infty$, $(-g_1)\to G^{(1)}$ of equation (23).

If in equation (A.1) we sum over n using Poisson's summation formula (Lighthill 1958) before doing the (κ, θ) integration, we get

$$-g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = \frac{1}{2\pi}\int_0^\infty \frac{\kappa\,\mathrm{d}\kappa J_0(\kappa\rho)\exp(KL)}{K\{\exp(2KL)-1\}}$$
$$\times[\cosh\{K(|z-z'|-L)\}-\cosh\{K(|z+z'|-L)\}], \quad \text{(A.3)}$$

where $K = (\kappa^2+\xi^2/c^2)^{1/2}$. Using the identity,

$$\int_0^\infty \frac{\kappa\,\mathrm{d}\kappa J_0(\kappa\rho)}{\{\kappa^2+(\xi^2/c^2)\}^{1/2}}\exp[-|z-z'|\{\kappa^2+(\xi^2/c^2)\}^{1/2}] = \frac{\exp(-\xi R/c)}{R}, \quad \text{(A.4)}$$

where $R = \{(z-z')^2+\rho^2\}^{1/2}$, equation (A.3) can be re-arranged to give,

$$-g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = \frac{1}{4\pi}\left(\frac{\exp(-\xi R/c)}{R}+2\int_0^\infty \frac{\kappa\,\mathrm{d}\kappa J_0(\kappa\rho)}{K\{\exp(2KL)-1\}}\right.$$
$$\left.\times[\cosh\{K(z-z')\}-\cosh\{K(z+z'-L)\}\exp(KL)]\right). \quad \text{(A.5)}$$

In this form the free-space Green function is exhibited explicitly together with the correction term arising out of the boundary effects.

The results of equation (26) are obtained from the non-retarding form of equation (A.5), with the substitution $2L\kappa = t$,

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi}\left[\frac{1}{R}+\frac{1}{L}\int_0^\infty \frac{\mathrm{d}t\, J_0(\rho t/2L)}{\exp(t)-1} \times \left\{\cosh\left(\frac{(z-z')t}{2L}\right)-\cosh\left(\frac{(z+z'-L)t}{2L}\right)\exp\left(\frac{t}{2}\right)\right\}\right]. \quad \text{(A.6)}$$

Since the main contribution to the integral comes from the neighbourhood of $t \to 0$, the Bessel function and cosh function are expanded in power series and integrated term by term to give us,

$$\begin{aligned} G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') &= \frac{1}{4\pi}\left[\frac{1}{R}+\frac{2}{L}\left\{-\ln 2+\left(\frac{(z-z')^2}{8L^2}-\frac{\rho^2}{16L^2}\right)\xi(3) - \left(\frac{(z+z'-L)^2}{8L^2}-\frac{\rho^2}{16L^2}\right)\zeta(3,\tfrac{1}{2})+\mathrm{O}\left(\frac{z-z'}{L}\right)^4\right\}\right] \\ &= \frac{1}{4\pi}\left(\frac{1}{R}-\frac{2\ln 2}{L}+\frac{\zeta(3)}{4L^3}\{3\rho^2-7(z+z'-L)^2+(z-z')^2\} + \dots\right). \end{aligned} \quad \text{(A.7)}$$

Here $\zeta(3,\frac{1}{2})$ is a generalized zeta function (Whittaker and Watson 1952) satisfying the relation $\zeta(s,\frac{1}{2}) = (2^s-1)\zeta(s)$. The diadic $(\nabla\nabla')G^{(1)}(\boldsymbol{r},\boldsymbol{r}')$ is then constructed trivially. This leads to the result of equation (26*a*). For the results of equations (26*b*) and (26*c*) the starting point is again equation (A.6) in the following modified forms:

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi}\left[\frac{1}{R}-\frac{1}{R_{(+)}}+\frac{1}{L}\int_0^\infty \frac{\mathrm{d}tJ_0(\rho t/2L)}{\exp(t)-1}\left\{\cosh\left(\frac{(z-z')t}{2L}\right)-\cosh\left(\frac{(z+z')t}{2L}\right)\right\}\right], \quad \text{(A.8)}$$

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi L}\int_0^\infty \frac{\mathrm{d}tJ_0(\rho t/2L)\exp(t/2)}{\exp(t)-1}\left\{\cosh\left(\frac{(|z-z'|-L)t}{2L}\right)-\cosh\left(\frac{(|z+z'|-L)t}{2L}\right)\right\} \quad \text{(A.9)}$$

where $R_{(+)} = \{(z+z')^2+\rho^2\}^{1/2}$. The rest of the procedure is similar, involving expansions of the Bessel and cosh functions.

To obtain the results for $R \ll L$ in the retarded case, it is convenient to start with $(-g_1)$ in the form,

$$-g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = \frac{1}{4\pi}\sum_{l=-\infty}^{\infty}\left(\frac{\exp(-R_l\xi/c)}{R_l}-\frac{\exp(-R_l'\xi/c)}{R_l'}\right), \quad \text{(A.10)}$$

which can be obtained from equation (27) by using Poisson's summation formula. Here, $R_l^2 = (2lL+|z-z'|)^2+\rho^2$, and $R_l'^2 = (2lL+|z+z'|)^2+\rho^2$.

One can use the formula (Watson 1958),

$$\frac{\exp\{-\eta(t^2+s^2-2ts\cos\theta)^{1/2}\}}{(t^2+s^2-2ts\cos\theta)^{1/2}} = \sum_{l'=0}^{\infty}\frac{(2l'+1)}{\sqrt{(ts)}}K_{l'+\frac{1}{2}}(t\eta)I_{l'+\frac{1}{2}}(s\eta)P_{l'}(\cos\theta); \quad t > s, \quad \text{(A.11)}$$

to write,

$$\sum_{l=-\infty}^{\infty} \frac{\exp(-R_l\xi/c)}{R_l} = \frac{\exp(-\xi R/c)}{R} + 2\sum_{l'=0}^{\infty}(4l'+1)\sum_{l=1}^{\infty}\frac{1}{\sqrt{(2lLR)}} \times K_{2l'+\frac{1}{2}}(2lL\xi/c)I_{2l'+\frac{1}{2}}(\xi R/c)P_{2l'}\left(\frac{|z-z'|}{R}\right). \tag{A.12}$$

This expansion is permissible since $2lL \gg R, (l \neq 0)$. To expand the second sum over $\{\exp(-\xi R_l'/c)/R_l'\}$, we write

$$z+z' = 2L\beta - \delta; \qquad 0 < \beta < 1, \qquad \delta = O\left(\frac{1}{L}\right). \tag{A.13}$$

Then,

$$R_l'^2 = \{2L(l+\beta)\}^2 + 2L(l+\beta)(R'')\left(\frac{z+z'-2L\beta}{R''}\right) + (R'')^2,$$

where $(R'')^2 = (z+z'-2L\beta)^2 + \rho^2$. Since $R'' \ll 2L(l+\beta)$, we can use equation (A.11) to write

$$\sum_{l=-\infty}^{\infty}\frac{\exp(-R_l'\xi/c)}{R_l'} = \sum_{l'=0}^{\infty}(2l'+1)\sum_{l=1}^{\infty} \times\left(\frac{K_{l'+\frac{1}{2}}\{2L\xi(l-\beta)/c\}}{\{2L(l-\beta)\}^{1/2}} + (-1)^{l'}\frac{K_{l'+\frac{1}{2}}\{2L\xi(l+\beta-1)/c\}}{\{2L(l+\beta-1)\}^{1/2}}\right) \times\frac{I_{l'+\frac{1}{2}}(\xi R''/c)}{\sqrt{R''}}P_{l'}\left(\frac{z+z'-2L\beta}{R''}\right). \tag{A.14}$$

We can now use equations (A.12) and (A.14) in equation (A.10) to write,

$$g_1(\boldsymbol{r}, \boldsymbol{r}'; -\mathrm{i}\xi) = g_1^{(0)} + g_1^{(1)}; \qquad g_1^{(0)} = -\frac{\exp(\xi R/c)}{4\pi R}. \tag{A.15}$$

Here $g_1^{(1)}$ is the correction to the free-space Green function.

After construction of the diadic of equation (21), the integration over ξ in equation (10) can be done by expanding the Bessel functions $I_{l'+\frac{1}{2}}(R\xi/c)$, $I_{l'+\frac{1}{2}}(R''\xi/c)$ in powers of $(\xi R/c)$, $(\xi R''/c)$ and retaining terms of order $\{(\xi R/2c)^{5/2}/\sqrt{R}\}$, $\{(\xi R''/2c)^{5/2}/\sqrt{R}\}$.

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